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EXPLICIT CRACK PROBLEM SOLUTIONS OF HYBRID COMPOSITES

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Abstract—This paper examines the stress concentration problem in an idealized unidirectional continuous intraply hybrid fiber composite due to transverse multifilament failure. The boundary value problems are formulated based upon a two-dimensional elastic shear-lag model for a hybrid composite.

Two types of crack in a brittle two-phase hybrid composite are analysed. By assuming that a crack extends when the stress in the first unbroken fiber reaches the average fiber strength, it is possible to predict when each type of crack will extend. A crack in the weak phase begins to spread when the stress in the material exceeds the critical stress. The growth of the crack is not accompanied by breakage of strong fibers. A developed crack, which breaks the material completely, spreads when the stress-intensity factor reaches the critical value. The analytical solution of boundary value problems for two types of semi-infinite cracks allows the critical stress intensity factor and critical stress to be estimated.

1. INTRODUCTION

The resistance of composite materials to tensile fracture can be considerably improved using reinforcement by two or more types of strong fibers. Common fibers used in hybrid composites are Kevlar® aramid, boron, carbon or glass. Either thermosetting or thermoplastic matrices are applied. The applications and material properties of hybrid composites are surveyed in Kretsis (1987). The uncertainty of reliable experimental data and lack of knowledge about fracture processes necessitates the microstructural modeling of failure.

For uniaxial tension, it suffices to model a composite as a discrete system of reinforcing filaments, which are assumed to carry only tensile stress, and a matrix, which carries only shear stress (Cox, 1952). These assumptions are reasonable as long as the fiber component of the composite is much stiffer than the matrix component. This assumption considerably simplifies the problem, and is the basis of the shear-lag model of Hedgepeth (1961). Modifications of the primary idea were made to extend it to various fracture effects (Hedgepeth and Van Dyke, 1967; Zweben, 1974; Nairn, 1988a, b; Hikami and Chou, 1990; Dharani *et al.*, 1990).

Subsequent work has looked at the effect of matrix splitting (Van Dyke and Hedgepeth, 1969; Goree and Gross, 1979), matrix yielding and dynamic loading (Fichter, 1969), three-dimensional arrays (Van Dyke and Hedgepeth, 1969), transverse matrix stresses (Goree and Gross, 1979) and axial matrix stiffness (Kobelev, 1989).

The micromechanics of hybrid composite fracture was studied by Manders and Bader (1981a, b), Thorat and Lakkad (1983) and Fariborz and Harlow (1987). The shear-lag model is applied to the fracture problem of intraply hybrid composites by Fukuda and Chou (1983) and Fariborz *et al.* (1985). A hybrid composite consisting of two types of fibers arranged in alternating positions is studied by Zweben (1977), where the stress concentration factor and the ineffective length of a broken fiber are obtained. The stress redistribution for various arrangements of broken and unbroken filaments for a material with randomly arranged fibers is studied by Fariborz *et al.* (1985).

In the present study a hybrid shear model is presented to determine the failure conditions for a material, which consists of two types of periodically arranged fibers. The equations of the model and their dimensionless analogies are listed in the second section. The numerical results of fracture modeling are briefly described in the third section. The

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constitutive equations are written in a complex form and the problem is reduced to a boundary value problem in terms of transformations of the unknowns in section four. The fifth and sixth sections contain analytical solutions of the boundary value problem for cracks which propagate in weak or strong phases.

2. THE MODEL OF A HYBRID COMPOSITE

Consider a monolayer of thickness $\hbar$ of a hybrid fiber-reinforced composite (Fig. 1). The monolayer is formed periodically along the y axis with two types of fibers of unlimited length, arranged in alternating positions. The two types of fibers are characterized by the following properties: the first has Young's modulus E_a and the cross-sectional area f_a. The Young's modulus of the fibers of the second type is E_b and their cross-sectional area is f_b. The space between fibers is filled by a matrix with shear modulus G. Consider any fiber of the first type, and call it the central fiber. Let its number be i. The neighboring fiber of the second type to the left of the central fiber also has number i, while the neighboring fiber to the right has number $i+1$. The distance between the centers of the nearest fibers of the same type is denoted as ℓ.

Denote the axial forces in the fibers of the first and second type with the numbers i by t_i^0 and s_i^0. The axial displacements of the fibers are denoted by u_i^0 and v_i^0, respectively. The equilibrium equations of the material are written as

$$\frac{ds_i^0}{dx} = \tau_i^* - \tau_i, \quad \frac{dt_i^0}{dx} = \tau_i - \tau_{i+1}^*. \tag{1}$$

Here τ_i^*, τ_i are the shear stresses in the matrix to the left and right of the ith fiber of the second type. It is assumed that the behavior of a given filament is influenced only by its nearest neighbors. The shear stress between the fibers is thus assumed to be

$$\tau_i = G(u_i^0 - v_i^0)/\Delta, \quad \tau_i^* = G(v_i^0 - u_{i-1}^0)/\Delta. \tag{2}$$

The matrix thickness is denoted here by Δ. The tensile forces in the filament are related to their displacements by

$$t_i^0 = E_a f_a \frac{du_i^0}{dx}, \quad s_i^0 = E_b f_b \frac{dv_i^0}{dx}. \tag{3}$$

Introducing the dimensionless variables

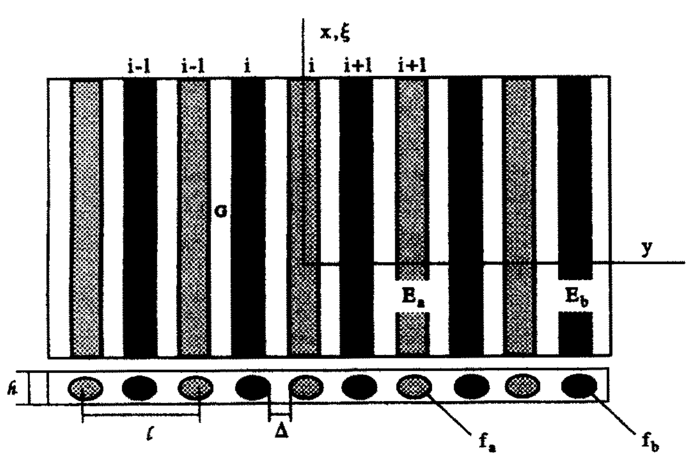

Fig. 1. Structure of hybrid monolayer.

$$u_i = u_i^0/\lambda, \quad v_i = v_i^0/\lambda, \quad \eta = x/\lambda, \quad t_i = t_i^0(E_a f_a)^{-1}, \quad s_i = s_i^0(E_b f_b)^{-1},$$

we rewrite (1)–(3) in the dimensionless form ($-\infty < i < \infty$):

$$\frac{1}{\kappa}\frac{\mathrm{d}^2 u_i(\eta)}{\mathrm{d}\eta^2} + v_i(\eta) - 2u_i(\eta) + v_{i+1}(\eta) = 0,$$

$$\frac{\mathrm{d}^2 v_i(\eta)}{\mathrm{d}\eta^2} + u_i(\eta) - 2v_i(\eta) + u_{i-1}(\eta) = 0, \quad t_i = \frac{\mathrm{d}u_i}{\mathrm{d}\eta}, \quad s_i = \kappa\frac{\mathrm{d}v_i}{\mathrm{d}\eta}. \tag{4}$$

Here λ is the character length of diffusion of the axial load in fibers (ineffective length) and κ the relative axial stiffness of fibers:

$$\lambda = \left(\frac{E_b f_b \Delta}{G}\right)^{1/2}, \quad \kappa = \frac{E_b f_b}{E_a f_a}.$$

As $\kappa \to \infty$ or $\kappa \to 0$ the model under consideration becomes the shear-lag model of a composite, containing only one type of fiber (Hedgepeth, 1961).

3. NUMERICAL MODELING—THE CRACK STRUCTURE

Consider now a material subjected to a homogeneous uniaxial tension at infinity. The boundary data correspond to the uniform far field condition, i.e. the axial deformation of each fiber at infinity is equal to a given constant ε. For hybrid composites it is well accepted that the effective elastic modulus $\bar{E}$ may be obtained from the rule of mixtures $\bar{E} = (E_a f_a + E_b f_b)/(\ell h)$. Thus, the effective stress at infinity is $\sigma^\infty = \bar{E}\varepsilon$. The uniform axial stresses in fibers of the first and second type are equal to $\sigma_a = E_a\sigma^\infty/\bar{E}$ and $\sigma_b = E_b\sigma^\infty/\bar{E}$, respectively. The fibers are assumed to be linearly elastic even at failure. Let the ultimate strength of the fibers of the first and second type be $\sigma_{\max a}$ and $\sigma_{\max b}$, respectively. Consider first, homogeneous materials, without imperfections, such as failures or flaws in fibers. As the effective stress increases, the failure of the fibers begins. Assume that the ultimate effective failure load for the fibers of the first type is more than that for the fibers of the second type. Accordingly, we call the fibers of the first type a strong phase, and the fibers of the second type a weak phase. It is easy to determine that the failure of the weak fibers occurs when the effective stress reaches the value $\sigma_1^\infty = \bar{E}\sigma_{\max b}/E_b$. The strength of the strong fibers is exhausted when the effective stress is $\sigma_2^\infty = \bar{E}\sigma_{\max a}/E_a$.

When the material contains dispersed flaws or local failures of fibers, the failure behavior of the material becomes complicated. As a rule, imperfections are distributed irregularly. Numerical modeling of the fracture reveals the following picture of failure of an idealized hybrid composite.

A local fiber failure leads to the increase of the stresses in the vicinity and provokes the failure of the weak phase. If the additional stress is too small to cause the damage of the strong phase, the process is not cumulative. A definite number of the weak fibers will be broken, and a cluster of damaged weak fibers will be formed. A further increase in the effective stress results in the following process: the clusters of failed weak fibers begin to spread in the direction normal to the fiber axes. The strong fibers bridge the sides of the cluster, and a kind of bridged flaw appears. It was found that for a sufficiently large number of fibers providing statistical homogeneity of the array, the effective stress, which corresponds to the onset of crack growth in the weak phase, has a definite value σ_b^∞, ($\sigma_b^\infty < \sigma_1^\infty$).

The second failure mechanism switches on when the applied load becomes sufficient to initiate propagation of a crack, which cuts the strong fibers. The stress state around the crack tip is singular, and it was observed that the material fails, when the magnitude of the stress intensity factor becomes equal to its critical value K_{cr}. The typical structure of a

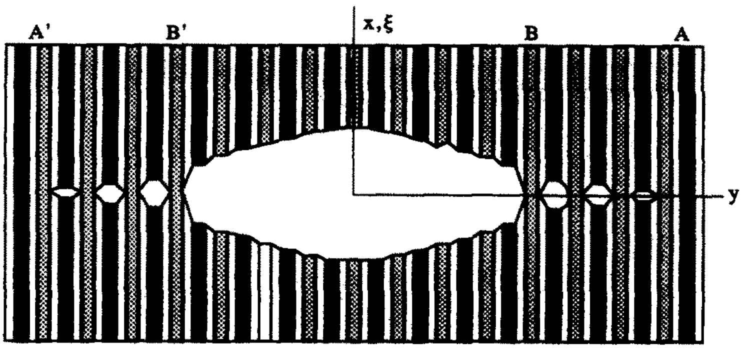

Fig. 2. Typical crack in hybrid monolayer. Computer simulation.

material near the crack tip is shown in Fig. 2. The weak phase is broken to the left of point B and is whole to the right. In the zone AB the strong fibers bridge the crack. The fibers also bridge the crack in zone *B′A′*. In zone *B′B* a developed crack appears, and here both phases are broken.

The aims of the subsequent consideration are:

(1) to determine the conditions of the crack growth in a weak phase, i.e. the stress σ_b^∞;
(2) to determine the stress intensity factor K_{cr}, corresponding to fracture initiation.

Note, that this purified approach ignores many details about damage initiation and propagation of the crack, but it serves mostly as an unsophisticated model of a hybrid material. Particularly, the statistical variation in fiber strength is ignored and the fracture conditions under which cracks might extend in composites are simplified. The present analysis can also be generalized to include a crack, splitting the material parallel to the fibers or propagating in another direction.

4. THE COMPLEX FORM OF THE CONSTITUTIVE EQUATIONS

We introduce the transformations:

$$U(w,\eta)=\sum_{n=-\infty}^{\infty}u_n(\eta)w^n,\quad V(w,\eta)=\sum_{n=-\infty}^{\infty}v_n(\eta)w^n,$$

$$T(w,\eta)=\sum_{n=-\infty}^{\infty}t_n(\eta)w^n,\quad S(w,\eta)=\sum_{n=-\infty}^{\infty}s_n(\eta)w^n,\quad |w|=1. \tag{5}$$

Everywhere we use w to denote the complex unit-circle parameter $w=\exp(\mathrm{i}\xi)$. The Laurent–Fourier series (5) is assumed to be absolutely convergent. Equations (4) in terms of transformations are equivalent to the system:

$$\frac{1}{\kappa}\frac{\partial^2 U}{\partial\eta^2}+\bar{\omega}V-2U=0,\quad \frac{\partial^2 V}{\partial\eta^2}+\omega U-2V=0, \tag{6}$$

where $\omega=1+w$, $\bar{\omega}=1+w^{-1}$. From (6) we have

$$\mathfrak{L}U=0,\quad \mathfrak{L}V=0,\quad \mathfrak{L}=\frac{\partial^4}{\partial\eta^4}-2(1+\kappa)\frac{\partial^2}{\partial\eta^2}+\kappa R^2(w), \tag{7}$$

where $R^2(w)=2-w-w^{-1}=4\sin^2(\xi/2)$.

Consider the solutions of eqns (7) in the lower half-plane, namely $-\infty<\eta<0$. The solutions of eqn (7), which are bounded in the limit $\eta\to-\infty$, have the form

$$V(w,\eta) = A_1(w)\exp[k_1(w)\eta] + A_2(w)\exp[k_2(w)\eta]. \tag{8}$$

The roots of the characteristic equation

$$k^4 - 2(1+\kappa)k^2 + \kappa R^2(w) = 0 \tag{9}$$

are denoted by $k_1(w)$ and $k_2(w)$:

$$k_1^2(w) = 1+\kappa+\sqrt{(1+\kappa)^2 - \kappa R^2(w)},$$
$$k_2^2(w) = 1+\kappa-\sqrt{(1+\kappa)^2 - \kappa R^2(w)}.$$

To simplify the notation, we adopt hereafter the following rule. If the argument η is omitted from the functions that depend explicitly on η, we can assume that we are considering the value of this function at $\eta = 0$, such that

$$t_n = t_n(0), \quad s_n = s_n(0), \quad T(w) = T(w,0), \quad S(w) = S(w,0),$$
$$u_n = u_n(0), \quad v_n = v_n(0), \quad U(w) = U(w,0), \quad V(w) = V(w,0).$$

The inverse of transformations (5) are

$$u_n(\eta) = \frac{1}{2i\pi}\oint_{|w|=1} U(w,\eta)\frac{\mathrm{d}w}{w^{n+1}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\mathrm{e}^{i\xi},\eta)\,\mathrm{e}^{-i\xi n}\,\mathrm{d}\xi,$$
$$v_n(\eta) = \frac{1}{2i\pi}\oint_{|w|=1} V(w,\eta)\frac{\mathrm{d}w}{w^{n+1}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\mathrm{e}^{i\xi},\eta)\,\mathrm{e}^{-i\xi n}\,\mathrm{d}\xi, \tag{10}$$

with similar expressions for $t_n(\eta)$ and $s_n(\eta)$. Substitution of (8) into (10) gives

$$v_n \equiv v_n(0) = \frac{1}{2i\pi}\oint_{|w|=1} [A_1(w)+A_2(w)]\frac{\mathrm{d}w}{w^{n+1}},$$
$$s_n \equiv \kappa\frac{\mathrm{d}v_n(\eta)}{\mathrm{d}\eta}\bigg|_{\eta=0} = \frac{\kappa}{2i\pi}\oint_{|w|=1} [k_1(w)A_1(w)+k_2(w)A_2(w)]\frac{\mathrm{d}w}{w^{n+1}}. \tag{11}$$

Using (6), one can express the functions $u_n(\eta)$ and $t_n(\eta)$ in terms of $A_1(w)$ and $A_2(w)$. We have

$$u_n(\eta)+u_{n-1}(\eta) = \frac{1}{2i\pi}\oint_{|w|=1} (1+w)U(w,\eta)\frac{\mathrm{d}w}{w^{n+1}}$$
$$= \frac{1}{2i\pi}\oint_{|w|=1}\left(2V - \frac{\partial^2 V}{\partial\eta^2}\right)\frac{\mathrm{d}w}{w^{n+1}}. \tag{12}$$

The differentiation of this identity gives

$$t_n(\eta)+t_{n-1}(\eta) = \frac{\mathrm{d}u_n(\eta)}{\mathrm{d}\eta} + \frac{\mathrm{d}u_{n-1}(\eta)}{\mathrm{d}\eta}$$
$$= \frac{1}{2i\pi}\oint_{|w|=1}(1+w)\frac{\partial U(w,\eta)}{\partial\eta}\frac{\mathrm{d}w}{w^{n+1}} = \frac{1}{2i\pi}\oint_{|w|=1}\left(2\frac{\partial V}{\partial\eta} - \frac{\partial^3 V}{\partial\eta^3}\right)\frac{\mathrm{d}w}{w^{n+1}}. \tag{13}$$

Particularly, for the displacements and tensile forces at $\eta = 0$ we have

$$u_n+u_{n-1} = \frac{1}{2i\pi}\oint_{|w|=1}\{[2-k_1^2(w)]A_1(w)+[2-k_2^2(w)]A_2(w)\}\frac{\mathrm{d}w}{w^{n+1}},$$

$$t_n+t_{n-1} = \frac{1}{2i\pi}\oint_{|w|=1}\{[2-k_1^2(w)]k_1(w)A_1(w)+[2-k_2^2(w)]k_2(w)A_2(w)\}\frac{\mathrm{d}w}{w^{n+1}}. \quad (14)$$

In terms of transformations $S(w)$, $T(w)$, $U(w)$ and $V(w)$ the expressions (11) and (14) can be written as

$$\begin{aligned} V(w) &= A_1(w)+A_2(w), \\ S(w) &= \kappa k_1(w)A_1(w)+\kappa k_2(w)A_2(w), \\ (1+w)U(w) &= [2-k_1^2(w)]A_1(w)+[2-k_2^2(w)]A_2(w), \\ (1+w)T(w) &= k_1(w)[2-k_1^2(w)]A_1(w)+k_2(w)[2-k_2^2(w)]A_2(w). \end{aligned} \quad (15)$$

Equations (15) express four unknowns in terms of only two functions, $A_1(w)$ and $A_2(w)$. The linear system (15) need not be solvable because the equations may be inconsistent. The following criterion tells us whether or not a solution exists. Rewrite the simultaneous equations (15) in the matrix form:

$$\mathbf{MX} = \mathbf{Y}, \quad (16)$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ \kappa k_1 & \kappa k_2 \\ 2-k_1^2 & 2-k_2^2 \\ k_1(2-k_1^2) & k_2(2-k_2^2) \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} A_1(w) \\ A_2(w) \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} V(w) \\ S(w) \\ (1+w)U(w) \\ (1+w)T(w) \end{bmatrix}.$$

The question about solvability is answered by the following statement: The system $\mathbf{MX} = \mathbf{Y}$ is solvable if and only if the vector $\mathbf{Y}$ has the property that $\mathbf{ZY} = 0$ whenever $\mathbf{ZM} = 0$.

The general solution of the associated homogeneous system $\mathbf{ZM} = 0$ is of the form $\mathbf{Z} = c_1\mathbf{Z}_1+c_2\mathbf{Z}_2$, where c_1, c_2 are arbitrary scalars, and $\mathbf{Z}_1, \mathbf{Z}_2$ is a basic set of solutions of the system:

$$\mathbf{Z}_1 = \begin{bmatrix} (2+k_1k_2)\kappa \\ -(k_1+k_2) \\ \kappa \\ 0 \end{bmatrix}, \quad \mathbf{Z}_2 = \begin{bmatrix} 0 \\ (2-k_1^2)(2-k_1^2) \\ \kappa k_1k_2(k_1+k_2) \\ -\kappa(2+k_1k_2) \end{bmatrix}.$$

Hence we enunciate the criterion of solvability of system (16) in the form:

$$\kappa(2+k_1k_2)V(w)-(k_1+k_2)S(w)+\kappa(1+w)U(w) = 0,$$

$$(2-k_1^2)(2-k_2^2)S(w)+(1+w)\kappa k_1k_2(k_1+k_2)U(w)-\kappa(2+k_1k_2)(1+w)T(w) = 0. \quad (17)$$

Equations (17) link the Laurent series of the unknown displacements and tensions in fibers at $\eta = 0$. System (17) presents the central equations for the theory because the different boundary value problems for the half-plane can be written in a similar form.

One short remark concerns the boundary conditions. The conditions of uniform far field require that the tensile forces at infinity in fibers are as follows:

$$s_n(\eta \to -\infty) = q_n^*, \quad t_n(\eta \to -\infty) = p_n^*,$$

where $p_n^* = \varepsilon$, $q_n^* = \varepsilon\kappa$.

For the resulting solution it is convenient to remove the stresses at infinity, placing the load on the crack surfaces. This can be achieved by applying the superposition principle. Due to the linearity of the material, add the uniform compression field such that the resulting axial deformation of the fibers at infinity vanishes. Correspondingly, this results in a uniform loading which is applied at the crack boundaries.

5. SEMI-INFINITE CRACKS IN THE WEAK PHASE—FLAW BRIDGING

Consider now a semi-infinite crack in the weak phase (Fig. 3). The fibers of the first type are broken to the left of point A and are whole to the right. The fibers of the second type constrict the crack edges. The boundary data for the problem under consideration are the following:

$$\begin{aligned} s_n &= -q_n^*, \quad u_n = 0, \quad \text{if} \quad n < 0, \\ v_n &= 0, \qquad\ \ u_n = 0, \quad \text{if} \quad n \geqslant 0. \end{aligned} \tag{18}$$

The fiber stresses vanish at infinity.

Any Laurent series can be split into two parts: the *principal* and the *regular* ones. The subscript "+" anywhere denotes an operator which correlates the Laurent series to its regular part and the subscript "−" denotes an operator which correlates the Laurent series to its principal part. So, for any Laurent series:

$$\mu(w) = \sum_{n=-\infty}^{\infty} \mu_n w^n, \quad \mu_+(w) = \sum_{n=0}^{\infty} \mu_n w^n, \quad \mu_-(w) = \sum_{n=-\infty}^{-1} \mu_n w^n. \tag{19}$$

In terms of Laurent series the boundary conditions (18) are written as:

$$U_+(w) = U_-(w) = V_+(w) = 0, \quad S_-(w) = Q_-(w), \tag{20}$$

where

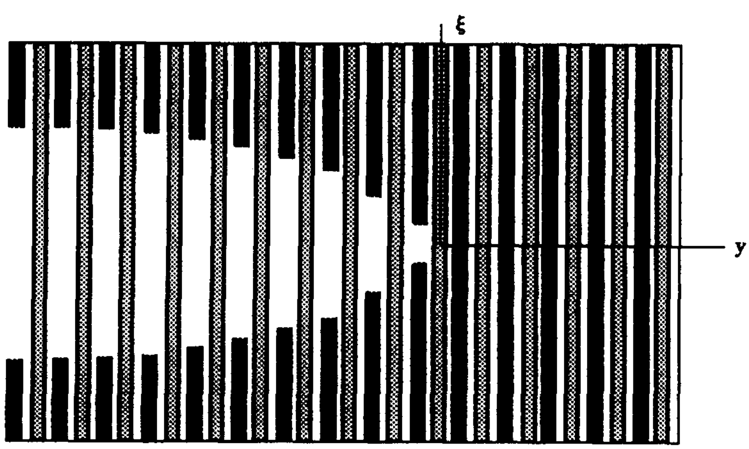

Fig. 3. The tip of semi-infinite crack in weak phase.

$$Q_-(w) = -\sum_{n=-\infty}^{-1} q_n^* w^n. \tag{21}$$

When the boundary conditions (20) are taken into account, (17) reduce to the Riemann–Hilbert boundary value problem (RHP):

$$\Phi(w)V_-(w) = S_+(w)+Q_-(w), \quad |w| = 1,$$
$$\Phi(w) = \kappa\frac{2+k_1k_2}{k_1+k_2} > 0, \quad \mathrm{Im}\,\Phi(w) = 0, \quad \mathrm{Ind}\,\Phi(w) = 0, \tag{22}$$

where

$$k_1k_2 = \kappa^{1/2}|R(w)|,$$
$$k_1+k_2 = \sqrt{2}\sqrt{1+\kappa^{1/2}|R(w)|+\kappa},$$
$$|R(w)| = \sqrt{2-2\,\mathrm{Re}\,(w)}.$$

The RHP (22) can be transformed to

$$\Phi_-(w)V_-(w)+\Psi_-(w) = S_+(w)/\Phi_+(w)+\Psi_+(w) = N(w),$$

where $\Phi(w) = \Phi_+(w)\Phi_-(w)$, and the factors of $\Phi(w)$ are

$$\Phi_+(z) = \exp\left(\frac{1}{2i\pi}\oint_{|w|=1} \ln \Phi(w)\frac{\mathrm{d}w}{w-z}\right), \quad |z| < 1,$$

$$\Phi_-(z) = \exp\left(-\frac{1}{2i\pi}\oint_{|w|=1} \ln \Phi(w)\frac{\mathrm{d}w}{w-z}\right), \quad |z| > 1.$$

The auxiliary function $\Psi(z)$ is defined as

$$\Psi(z) = \frac{1}{2i\pi}\oint_{|w|=1}\frac{Q_-(w)}{\Phi_+(w)}\frac{\mathrm{d}w}{w-z}. \tag{23}$$

The function $\Psi_+(z)$ is analytical in $|z| < 1$, and $\Psi_-(z)$ is analytical in $|z| > 1$, and $\Psi_+(w)-\Psi_-(w) = Q_-(w)/\Phi_+(w)$ on $|w| = 1$.

The function $N(w)$ is holomorphic in the entire plane, except at the point at infinity where it may have a pole of order not greater than n, and, by the generalized Liouville theorem, is a polynomial of degree n with arbitrary complex coefficients. The general solution of the problem is given by:

$$\Phi_-(z)V_-(z)+\Psi_-(z) = N(z), \quad |z| > 1,$$
$$S_+(z)/\Phi_+(z)+\Psi_+(z) = N(z), \quad |z| < 1,$$

and, consequently,

$$V_-(z) = [N(z)-\Psi_-(z)]/\Phi_-(z), \quad |z| > 1,$$
$$S_+(z) = [N(z)-\Psi_+(z)]\Phi_+(z), \quad |z| < 1.$$

The solution, vanishes at infinity, is unique and has the form:

$$V_-(z) = -\Psi_-(z)/\Phi_-(z), \quad |z| > 1,$$
$$S_+(z) = -\Psi_+(z)\Phi_+(z), \quad |z| < 1. \tag{24}$$

To formulate the fracture condition we need the expression for the force s_0 in the first unbroken fiber. Using (24) we have:

$$s_0 \equiv \frac{1}{2i\pi}\oint_{|w|=1} S_+(w)\frac{\mathrm{d}w}{w} = S_+(0) = -\Psi_+(0)\Phi_+(0). \tag{25}$$

For some special cases the integrals can be expressed in a closed form. Let the load on the boundaries of the crack have a power-law distribution $q_n^* = \sigma^{-n}$, where σ is a constant, $0 < \sigma < 1$. For a unit constant load which corresponds to uniform stress at infinity, the solution is found in the limit $\sigma \to 1$. In this case (21) assumes the form

$$Q_-(w) = (1-w/\sigma)^{-1}.$$

Integral (23) can be determined using Caushy's residue theorem $\Psi_+(0) = 1/\Phi_+(0) - 1/\Phi_+(\sigma)$ and (24) reduces to $s_0 = \Phi_+(0)/\Phi_+(\sigma) - 1$. For the practically important limit case $\sigma \to 1$ (constant load), we obtain

$$s_0 = \gamma(\kappa) - 1, \quad \gamma(\kappa) = \Phi_+(0)/\Phi_+(1). \tag{26}$$

The value $\Phi_+(0)$ is given by the formula:

$$\ln \Phi_+(0) = \frac{1}{2i\pi}\oint_{|w|=1} \ln \Phi(w)\frac{\mathrm{d}w}{w} \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\xi)\,\mathrm{d}\xi$$
$$= \ln\left(\kappa\sqrt{2}\right) + \frac{2}{\pi}J(\kappa^{1/2}) - \frac{1}{\pi}I(\kappa^{1/2}), \tag{27}$$

where

$$\varphi(\xi) = \ln\left[\kappa\sqrt{2}\left(1+\sqrt{\kappa}\left|\sin\frac{\xi}{2}\right|\right)\left(1+2\sqrt{\kappa}\left|\sin\frac{\xi}{2}\right|+\kappa\right)^{-1/2}\right],$$

$$J(x) = \int_0^{\pi/2} \ln(1+x\sin\xi)\,\mathrm{d}\xi, \quad I(x) = \int_0^{\pi/2} \ln(1+2x\sin\xi + x^2)\,\mathrm{d}\xi.$$

There is an algebraic relation between the integrals $J(x)$ and $I(x)$:

$$J(x) = I(y) - \frac{\pi}{2}\ln(1+y^2), \quad x = \frac{2y}{(1+y^2)}.$$

Thus,

$$I(\kappa^{1/2}) = J\left[\frac{1-\sqrt{1-\kappa}}{\kappa^{1/2}}\right] + \frac{\pi}{2}\ln(1+\kappa).$$

Now evaluate $\ln \Phi_+(1)$:

$$\ln \Phi_+(1) = \frac{1}{2i\pi}\oint_{|w|=1} \ln \Phi(w)\frac{\mathrm{d}w}{w-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\xi)\frac{\mathrm{d}\xi}{2} + \frac{1}{2\pi i}\int_{-\pi}^{\pi} \varphi(\xi)\cot\frac{\xi}{2}\frac{\mathrm{d}\xi}{2}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\xi)\frac{\mathrm{d}\xi}{2} + \frac{1}{2\pi i}\int_{-\pi}^{\pi} \varphi(\xi)\frac{\mathrm{d}\xi}{\xi} = \tfrac{1}{2}\ln \Phi_+(0) + \tfrac{1}{2}\varphi(0), \tag{28}$$

because, by the Plemelj theorem (Muskhelishvili, 1975)

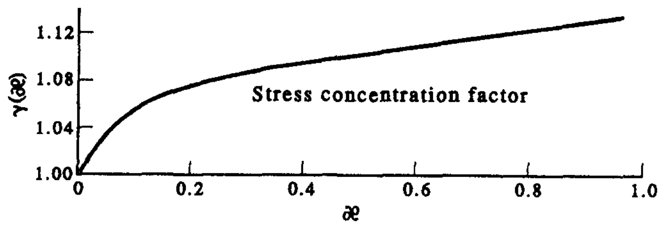

Fig. 4. Stress concentration factor $\gamma = \gamma(\kappa)$.

$$\int_{-\pi}^{\pi} \varphi(\xi)\frac{d\xi}{\xi} = i\pi\varphi(0).$$

Substitution of (27) and (28) into (26) leads to the following expression:

$$s_0 = \gamma(\kappa)-1, \quad \text{where} \quad \gamma(\kappa) = \exp\left[\frac{1}{\pi}J(\kappa^{1/2}) - \frac{1}{2\pi}J\left(\frac{1-\sqrt{1-\kappa}}{\kappa^{1/2}}\right)\right]. \tag{29}$$

We thus find that if the unit local forces are applied on the broken fibers at the crack boundaries and the material is unstressed at infinity, then the tension of the first fiber is equal to s_0. Now it is easy to calculate the stress in the first unbroken fiber for the uniform far field condition. Adding the far field to remove the boundary load, we obtain an expression for the maximal stress at the first fiber

$$\sigma_b = \gamma(\kappa)\varepsilon E_b = \gamma(\kappa)E_b\sigma^{\infty}/\bar{E}. \tag{30}$$

This reveals the physical significance of the function $\gamma(\kappa)$. It is equal to the *stress concentration factor* (SCF) for an intact fiber adjacent to an array of broken fibers. The dependence of $\gamma(\kappa)$ upon parameter κ is shown in Fig. 4.

Suppose, that the fibers break when the tensile stress in the first fiber σ_b reaches its ultimate value $\sigma_{b\max}$. Expression (30) gives that the limiting stress at infinity in the material, which corresponds to the onset of the crack growth in the weak phase, is equal to

$$\sigma_b^{\infty} = \frac{\bar{E}}{\gamma(\kappa)E_b}\sigma_{b\max}. \tag{31}$$

6. SEMI-INFINITE CRACK IN MATERIAL

A developed crack in a material is one which breaks both phases. The strong fibers are whole to the right of point B (Fig. 5), and broken to the left of it; the weak fibers are

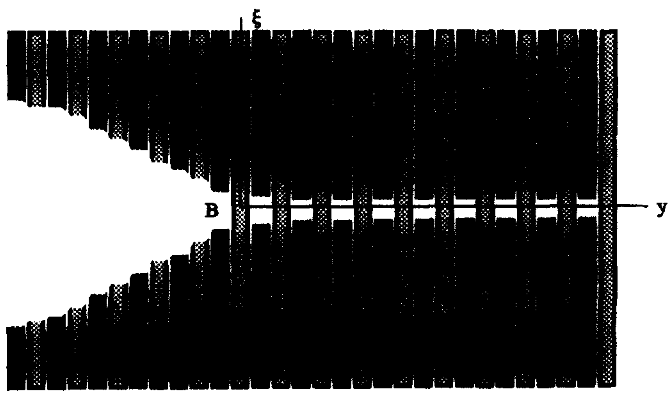

Fig. 5. The tip of semi-infinite developed crack.

broken both to the right and left of the outermost fiber. The conditions of propagation of the developed crack are given by the solution of the problem for a semi-infinite developed crack.

The superposition principle is applied again for the solution of the problem. Due to the linearity of the composite, add the uniform compression field such that the resulting axial deformation of the fibers at infinity vanishes. A uniform load is now applied at the crack boundaries.

The boundary data for the considered problem are the following:

$$s_n = -q_n^*, \quad t_n = -p_n^*, \quad \text{if} \quad n < 0,$$
$$s_n = -q_n^*, \quad u_n = 0, \quad \text{if} \quad n \geqslant 0. \tag{32}$$

The fiber stresses vanish at infinity. In terms of Laurent series the boundary conditions (32) are written as:

$$S_+(w) = Q_+(w), \quad S_-(w) = Q_-(w),$$
$$U_+(w) = 0, \quad T_-(w) = P_-(w), \tag{33}$$

where

$$P_-(w) = -\sum_{n=-\infty}^{-1} p_n^* w^n, \quad Q_+(w) = -\sum_{n=0}^{\infty} q_n^* w^n, \quad Q_-(w) = -\sum_{n=-\infty}^{-1} q_n^* w^n. \tag{34}$$

When the boundary conditions (33) are taken into account, (17) reduce to the RHP:

$$\kappa^{3/2}|R(w)|\,U_-(w) = \Phi(w)[T_+(w)+P_-(w)]+\nu(w)[Q_+(w)+Q_-(w)], \quad \text{on} \quad |w| = 1, \tag{35}$$

where

$$|R(z)| = R_+(z)R_-(z),$$
$$R_+(z) = \sqrt{1-z}, \quad |z| < 1, \quad R_-(z) = \sqrt{1-z^{-1}}, \quad |z| > 1,$$
$$\nu(z) = \frac{(2-k_1^2)(2-k_2^2)}{(1+w)(k_1+k_2)} > 0, \quad \operatorname{Im} \nu(w) = 0 \quad \text{on} \quad |w| = 1.$$

The bounded contour classical solution of RHP (35) is

$$U_-(z) = [M(z)-\chi_-(z)]\Phi_-(z)\kappa^{-3/4}/R_-(z), \quad |z| > 1,$$
$$T_+(z) = [M(z)-\chi_+(z)]R_+(z)\kappa^{3/4}/\Phi_+(z), \quad |z| < 1,$$

where $M(z)$ is an arbitrary complex polynomial, and $\chi(z)$ is the auxiliary function

$$\chi(z) = \frac{\kappa^{-3/4}}{2i\pi}\oint_{|w|=1}\frac{\Phi_+(w)}{R_+(w)}\left[P_-(w)+\frac{\nu(w)Q(w)}{\Phi_-(w)}\right]\frac{\mathrm{d}w}{w-z}.$$

Note that $\chi_+(z)$ is the analytical function in $|z| < 1$, $\chi_-(z)$ is the analytical function in $|z| > 1$, and on the unit circle $|w| = 1$:

$$\chi_+(w)-\chi_-(w) = \frac{\Phi_+(w)}{R_+(w)}\left[P_-(w)+\frac{\nu(w)Q(w)}{\Phi_-(w)}\right].$$

The unique classic solution of RHP (35), which is bounded on the contour $|w| = 1$ and vanishes at infinity, is

$$U_-(z) = -\chi_-(z)\Phi_-(z)\kappa^{-3/4}/R_-(z), \quad |z| > 1,$$
$$T_+(z) = -\chi_+(z)R_+(z)\kappa^{3/4}/\Phi_+(z), \quad |z| < 1. \tag{36}$$

To formulate the fracture condition we need the expression for the force in the first unbroken strong fiber. Using (36) we have

$$t_0 \equiv \frac{1}{2i\pi}\oint_{|w|=1} T_+(w)\frac{dw}{w}$$
$$= -\frac{1}{2i\pi}\frac{R_+(0)}{\Phi_+(0)}\oint_{|w|=1}\frac{\Phi_+(w)}{R_+(w)}\left[P_-(w)+\frac{v(w)Q(w)}{\Phi_-(w)}\right]\frac{dw}{w}. \tag{37}$$

Consider now a power-law distribution $p_n^* = \mathcal{P}\tau^{-n}$, $q_n^* = \mathcal{Q}\tau^{-n}$, for $n < 0$ and $q_n^* = \mathcal{Q}\tau^n$ for $n > 0$. Here τ is a constant and $0 < \tau < 1$; $\mathcal{P}$ and $\mathcal{Q}$ are the intensities of loads, applied to the fibers of first and second kind respectively. For the power-law distribution the integrals can be expressed in a closed form. In this case (21) assumes the form:

$$P_-(w) = \mathcal{P}(1-w/\tau)^{-1}, \quad Q_-(w) = \mathcal{Q}(1-w/\tau)^{-1}, \quad Q_+(w) = \mathcal{Q}(w\tau-1)^{-1}, \tag{38}$$

and (37) reduces to

$$t_0 = \mathcal{P}\left[\frac{\Phi_+(\tau)}{\Phi_+(0)(1-\tau)^{1/2}}-1\right]+\mathcal{Q}\left[\frac{v(w)}{\Phi_+(0)\Phi_-(w)(1-\tau)^{1/2}}\right].$$

Function t_0 has the following asymptotic as $\tau \to 1$:

$$t_0 \propto \left[\mathcal{P}\frac{\Phi_+(\tau)}{\Phi_+(0)}+\mathcal{Q}\frac{v(w)}{\Phi_+(0)\Phi_-(w)}\right]\frac{1}{\sqrt{1-\tau}} \propto (\mathcal{P}+\mathcal{Q})\frac{\Phi_+(\tau)}{\Phi_+(0)}\frac{1}{\sqrt{1-\tau}}. \tag{39}$$

Thus, the stress in the first solid fiber is expressed in terms of parameters of microstructure and applied load. The next step is to establish a connection between the stress in the fiber and the stress-intensity factor.

Consider a functional which represents the stress-intensity factor of the exterior problem for a semi-infinite crack ($-\infty < x < 0$):

$$K = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sigma(-\xi)\,d\xi}{\sqrt{\xi}}. \tag{40}$$

Here $\sigma(x)$ is a load which is applied to the crack surfaces. In the case under consideration the load $\sigma(x)$ is a sum of partial loads, $p(x)$ and $q(x)$. These loads are applied to the broken ends of fibers of the first and the second kind, $\sigma(x) = p(x)+q(x)$, ($-\infty < x < 0$).

The sequences of concentrated forces with a power-law distribution are equivalent to the series with Dirac δ functions with the supports in regularly spaced points:

$$p(\xi) = \mathcal{P}\sum_{n=-\infty}^{-1}\tau^{-n}\delta(\xi+n\ell),$$
$$q(\xi) = \mathcal{Q}\sum_{n=-\infty}^{-1}\tau^{-n}\delta\left(\xi+n\ell-\frac{\ell}{2}\right)+\mathcal{Q}\sum_{n=0}^{\infty}\tau^{n}\delta\left(\xi+n\ell-\frac{\ell}{2}\right).$$

Substitution of the last expression in (40) gives:

$$K = \sqrt{\frac{2}{\pi\ell}}\left[\mathscr{P}\sum_{n=1}^{\infty}\frac{\tau^n}{\sqrt{n}} + Q\sum_{n=1}^{\infty}\frac{\tau^n}{\sqrt{n-1/2}}\right]$$

$$= \sqrt{\frac{2}{\pi\ell}}\tau[\mathscr{P}F(\tau, 1/2, 1) + \mathscr{Q}F(\tau, 1/2, 1/2)], \tag{41}$$

where $F(a, b, z)$ is a confluent hypergeometric function (Gradshteyn and Ryzlik, 1965). For a unit constant load which corresponds to uniform stress at infinity, the solution is found in the limit $\tau \to 1$. Using the formula (Gradshteyn and Ryzlik, 1965)

$$\lim_{\tau\to 1}\sqrt{1-\tau}F(\tau, 1/2, \alpha) = \Gamma(1/2) = \sqrt{\pi}, \quad \text{for} \quad \alpha = 1/2 \quad \text{and} \quad \alpha = 1,$$

we obtain the following asymptotic expansion (Erdelyi, 1956):

$$K \propto \sqrt{\frac{2}{\ell(1-\tau)}}(\mathscr{P} + \mathscr{Q}). \tag{42}$$

Comparing with (39) we find the relation between stress-intensity factor K and maximal tension in the fiber:

$$t_0 = \sqrt{\ell/2}K\gamma^{-1}(\kappa), \tag{43}$$

because $\lim_{\tau\to 1}[\Phi_+(\tau)/\Phi_+(0)] = \Phi_+(1)/\Phi_+(0) = 1/\gamma(\kappa)$.

It is remarkable that expression (43) contains the same coefficient $\gamma(\kappa)$ (stress concentration factor) which appeared in the problem of a semi-infinite crack in the weak phase [eqn (29)].

It is assumed that the strong fibers break when the maximum stress $\sigma_{\max a}$ is reached. From (43), it follows that the critical stress-intensity factor corresponding to the onset of the growth of the developed crack in the dimensioning variables is

$$K_{cr} = \gamma(\kappa)\sigma_{\max a}\frac{f_a}{h}\sqrt{\frac{2}{\ell}}. \tag{44}$$

That is, K_{cr}, which was derived from the stress-state solution at the crack tip, is a single parameter that describes that stress state. When the stress state reaches a critical level for this parameter, the material fails.

7. CONCLUSIONS

It is assumed that a brittle two-phase hybrid composite can have two main types of cracks. A crack in the weak phase begins to spread when the stress in the material exceeds the average value (30). A developed crack, which breaks the material completely, spreads when the stress-intensity factor exceeds the critical value (44). If initial flow size is L, a developed crack appears when the effective stress at infinity attains the value

$$\sigma_a^\infty = K_{cr}(\pi L)^{1/2},$$

provided that $\sigma_a^\infty < \sigma_b^\infty$. Otherwise, if $\sigma_a^\infty > \sigma_b^\infty$, when the tensile stress reaches the critical value (30), crack growth begins in the weak phase but is not accompanied by breakage of strong fibers.

The fracture analysis derived in this paper is specifically for brittle unidirectional composites, and cannot replace the techniques which can also handle multidirectional materials and nonlinear material behavior. But for those who want to optimize the material behavior of composites and need the data for characterizing the fracture toughness and material failure, this analysis provides a logical starting point.

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